PREDICTING RECESSIONS IN THE U.S. WITH YIELD CURVE SPREAD

A Thesis
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

Di Huang

In Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE

Major Department:
Statistics

April 2013

Fargo, North Dakota

North Dakota State University
Graduate School

Title

Predicting Recessions in the U.S. with Yield Curve Spread

By

Di Huang

The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

MASTER OF SCIENCE

SUPERVISORY COMMITTEE:

Dr. Gang Shen
Chair

Dr. Rhonda Magel

Dr. Seung Won Hyun

Dr. Ruilin Tian

Approved:

04/03/2013
Date

Rhonda Magel
Department Chair

ABSTRACT

This paper proposes a hidden Markov model for the signals of U.S. recessions. The predictors in model includes the spread of interest rate between 10-year Treasury bond and 3-month Treasury bill, the rate of real M2 growth, the change in the Standard and Poor's 500 index of stock prices, and the spread between the 6-month commercial paper and 6-month Treasury bill rates. Our model incorporates the temporal dependence between the recession signals and provides an estimate of the long-term probability of recessions. The empirical results indicate the hidden Markov model well predict the signal of recessions in the U.S.

ACKNOWLEDGMENTS

I take this opportunity to express my profound gratitude and deep regards to my advisor Dr. Gang Shen for his exemplary guidance on statistical modeling and model estimating, predicting, and coding in R program. Also, I want to thank my co-advisor Dr. Ruilin Tian for her meticulous assistance on econometric analysis throughout the course of this thesis. The support, help and guidance given by them time to time shall carry me a long way in the journey of life on which I am about to embark.

I also take this opportunity to express a deep sense of gratitude to my committee members Dr. Rhonda Magel and Dr. Seung Won Hyun for their cordial support and guidance, which helped me in completing this task through various stages.

Lastly, I would like to thank my family and friends for their constant support and encouragement without which this research would not be possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1. INTRODUCTION

The yield curve has been one dominant forecasting tool in predicting future economic growth and has fascinated economists for the past several decades. Traditionally, analysis using the yield curve spread focuses on the time series model and sets the dependent variable as growth rate of Gross Domestic Production (GDP). Recent studies, however, use a different statistical procedure, the so-called probit model, in predicting the binary variables representing recessions. No matter how the dependent variables change, the most useful predictor has been the yield curve spread.

Before we present the further statistical analysis, it is important to understand the meaning of recession. Defined by NBER, the national bureau of economic research, recession is a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales.

Recession is not what people want, because negative effects of recession not only affect the U.S. nation, but also average income people in multiple ways. For instance, a recession means a fall in GDP and national output, which will trigger a rise in unemployment. In the long term, the recession will affect the confidence of private investors. Therefore, it is necessary to study the probability of recession in order to prevent a crisis occurring in the future.

Typically, many researchers use the yield curve spread as a dominant predictor, because it contains useful information in forecasting recessions. By definition, the yield curve is a line that plots the interest rates of bonds having equal credit quality but differing maturity dates, for example, three-month, six-month, ten-year and thirty-year U.S. Treasury debt. In other words, the yield curve is a line graph of future bond rates of various durations. Yield curve spread refers to the difference between the long-term and short-term interest rates. Figure 1 illustrates three major shapes of yield curve spread: normal, flat, and inverted. In most cases, the shape is upward. However, the yield curve spread becomes troubling when the plot flattens, or even worse, inverts.

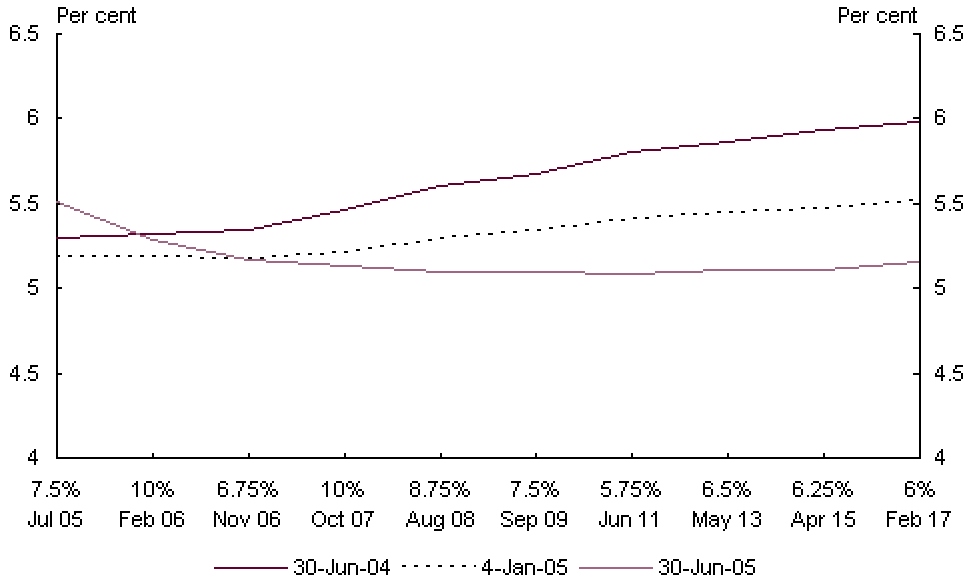

Figure 1: Normal, Flatten, and Invert Yield Curve Spread

Theoretically, there are at least three ways to explain that the spread of the yield curve contains information of recession, which are expectation theory, effects of monetary policies, and maximization of inter-temporal consumer choices. The expectation theory assumes that long-term interest rates reflect the expected path of future short-term interest rates. And for any choice of holding period, investors do not expect to realize different returns from holding bonds of different maturity dates. Thus anticipation of a recession implies an expectation of decline of future interest rates, which triggers the government to apply a counter-cyclical monetary policy designed to stimulate the economy. This policy is translated into a decrease of long-term interest rates.

Another way to explain the recession information contained in the yield curve spread is related to the effects of monetary policy. For instance, when monetary policy is tightened, short-term interest rates rise; long-term interest rates also typically rise, but usually by less than the current short rates, leading to a downward-sloping term structure. The third way is based on maximization of the inter-temporal consumer choices. The consumers prefer a stable level of income rather than very high income during expansion and very low income during slowdowns. Thus, if the consumers expect a reduction of their income during future recession, they prefer to save and buy long-term bonds in order to get payoffs in the slowdown. By doing that, they increase the demand for long-term bond, which leads to a decrease of the corresponding yield.

The effectiveness of the yield curve spread is tested using the probit model by Estrella and Mishkin (1996). The probit model predicts the recession dummy and concludes that the yield curve spread is the most powerful predictor. Dueker (1997) modifies the traditional probit model by adding a lagged dependent variable to the linear regression model and concludes that the yield curve spread remains the best recession predictor.

The shortcoming of previous studies is that they do not consider the dependent variable, recession, as a time series value with its own autocorrelation and far from normal distribution with a mean of zero. For instance, the occurrence of recession has some coherence. Whether recession happens in a current period is, to a great extent, relative to the economic status in prior periods.

In this project, we examine the power of the yield curve spread in predicting the signal of recession occurrence and take the dependency of time series value into consideration by building a hidden Markov model. To begin, we provide an introduction in Chapter 2. The literature review, and the methodology of the hidden Markov model are illustrated in Chapter 3. Chapter 4 focuses on a simulation study in order to estimate and compare the power of our hidden Markov model and the standard probit model. Chapter 5 is the data description part. The historical data analysis and model selection parts are presented in Chapter 6. Finally, Chapter 7 includes conclusions and discussion.

2. LITERATURE REVIEW

The study on the yield curve spread as a dominant predictor of the economy activities in U.S. did not attract attention of researchers until the late 1980s, when Stock and Watson (1989) examined combinations of fifty-five financial indicators in order to find out the economic variables that can best predict the future economic activity. Out of massive candidate variables, they limited their selection to seven variables including the yield curve spread representing the spread of the 10-year Treasury bond and 1-year Treasury bill yields.

In 1991, the yield curve spread was tested again by Estrella and Hardouvelies (1991). They examined the quarterly data from 1955 to 1988. The results suggested that the yield curve spread has more forecasting power in predicting the future economy than the lagged output growth, lagged inflation, the index of leading indicators, and the real short-term interest rates. Moreover, the yield curve spread is one of the predictors of the real Gross National Product (GNP), along with consumption, consumer durable, and investment.

A few years later, the yield curve spread, once again, been proved as one of the dominant predictors of future economy activities by Hu (1993) through examining growth rate of the real Gross Domestic Product (GDP). The result of the study supported that the slope of the yield curve spread has extra predict power over the lagged GDP growth, stock price, and inflation.

When most researchers focused on testing the forecasting power of explanatory variables to predict economic activity, an extraordinary breakthrough came. Different from previous studies, Estrella and Mishkin (1996) tested the ability of financial indicators on predicting recessions. They built a probit model in order to measure the probability of having recessions. They performed an out-of-sample test and examined quarterly data from the first quarter of 1960 to the first quarter of 1995. The yield spread used in the model was measured by the difference between the 10-year Treasury bond and 3-month Treasury bill. Besides, they involved other variables include: New York Stock Exchange (NYSE) price index, the leading index of Commerce Department, and lagged growth in real GDP.

The probit model is a wildly-used econometric model in which the dependent variable Y can be only 1 or 0. The basic assumption of the probit model is that the function follows a normal distribution. In the standard probit model proposed by Estrella and Mishkin (1996), the dependent variable R_t is a binary series that taking value either 1 or 0, where 1 indicates recession and 0 indicates no recession, respectively.

The probability of recession occurring at time t, with a forecast horizon of k is expressed as:

$$P(R_t = 1) = \Phi(c_0 + c_1 X_{t-k}) \tag{1}$$

In this equation, $\Phi(.)$ is the cumulative standard normal distribution function, X is an array that represents a set of financial factors which are powerfully affecting the economy, and k indicates the forecast horizon.

The log-likelihood function is given as follows:

$$L = \sum_t R_t \log(P(R_t = 1|X_{t-k})) + \sum_t (1 - R_t) \log(P(R_t = 0|X_{t-k})) \tag{2}$$

In the standard probit model proposed by Estrella and Mishkin (1996), pseudo $- R^2$ is used as a criteria in measuring goodness of fit.

$$\text{pseudo} - R^2 = 1 - \left(\frac{L_u}{L_c}\right)^{-(2/n)L_c} \tag{3}$$

In the pseudo $- R^2$ equation, the log-likelihood function of the probit model L_u is compared to L_c, the reduced log-likelihood function that only contains constant and free of explanatory variables. Since the L_u will always be greater than L_c, pseudo $- R^2$ generates a value between 0 and 1 with a larger value indicating a better fit. For instances, the closer the pesudo $- R^2$ value is to 1, the greater the explanatory power of the model.

In this probit model, the forecast horizons observed by Estrella and Mishkin (1996) are one, two, four, and six quarters. The result of probit model is the probability of having a recession ranges from 0 to 1 indicating 0% to 100%. The empirical results support that all variables have some forecasting ability one quarter ahead. The results suggest that the leading indicator index and the Stock-Watson index are the necessary predictors when the forecasting horizon is one quarter ahead. However, this situation changes when the horizon becomes two or more quarters. For instance, the leading indicator of Commerce Department's index misleads several recession signals during period from 1982 to 1990. The Stock-Watson index fails to predict the recession during the 1990-91 period. Compared to these variables, the yield curve spread is a stronger indictor in predicting the recession of 1990-91. However, the signal of 1990-91 recession is relatively weak with only 25%. On this basis, Estrella and Mishkin (1996) claim that 25% could be a threshold indicating that if the probability is greater than 25% a recession is signalled ahead, otherwise is not.

In 1997, Dueker (1997) re-examined the forecast power of yield curve spread to predict the signal of U.S. recession using the standard probit model proposed by Estrella and Mishkin (1996). Dueker (1997) modified the standard probit model on a few aspects. In the first place, Dueker (1997) tested five explanatory variables including the change in the leading indicators of Commerce Department, the real M1 growth, the change in the Standard and Poor's 500 index of stock prices, credit spread, and the yield curve spread. The quarterly data he tested were raging from January 1958 to May 1995. Secondly, in order to incorporate the temporal dependence between R_t and R_{t-k}, Dueker (1997) proposed a modified probit model which seemed somewhat unnatural.

$$P(R_t = 1) = \Phi(c_0 + c_1 X_{t-k} + c_2 R_{t-k}) \tag{4}$$

R is an unobserved latent variable with mean of zero. The forecast horizons are one, two, three, and four quarters ahead. Similar to Estrella and Mishkin (1996), Dueker (1997) used pseudo $- R^2$ to measure the performance of the variables. Specifically, L_u is from the full model, and L_c is from the reduced model.

The result of the empirical study supported that the model with lagged dependent variable has more ability in capturing the duration of recession than the standard probit model. However, it still fails to foresee the onsets of recessions in 1960, 1970, 1980 and 1990.

In 2005, Chauvet and Potter (2005) tested the ability of yield curve spread to predict U.S. recessions by applying several extensions of the probit model proposed by Estrella and Mishkin (1996). For instance, they compared the results of four different probit models which are: the time invariant conditionally independent version, the business cycle specific conditionally independent model, the time invariant probit model with autocorrelated errors, and the business cycle specific probit model with autocorrelated errors. The quarterly data they applied were from January 1955 to December 2000. Most of the models fail to predict the 1990 recession. Furthermore, the result after comparing indicates that the more complex models provide less strong and precise signals. In other words, the standard probit model with yield curve spread as explanatory variable provides the best result in predicting U.S. recessions.

In short summary, the yield curve is a useful predictor of recession, and the standard probit model prevails over the other complex probit models in forecasting. However, as we mentioned before, a potential problem of the probit model is that it does not consider dependence of the binary time series of recession signals. Therefore, in this research, instead of modifying the probit model, we propose a hidden Markov model in measuring the forecast ability of the yield curve spread.

3. HIDDEN MARKOV MODEL

Let X_t be the indicator of whether economy, at time t, is in recession or not.

Our hidden Markov model for X_t, t=1, 2, ..., is:

$$X_{t+1} = V_t X_t + (1 - V_t) Y_{t+1} \tag{5}$$

Where $\{Y_t\}$ is a sequence of independent Bernoulli, with $P(Y_t = 1) = \pi$.

π is the long-term probability of recessions.

$\{V_t\}$ is a sequence of independent Bernoulli, with $P(V_t = 1) = \theta_t$.

θ_t is explained by financial indicators at time t as:

$$\theta_t = \frac{exp^{\beta_0 + \beta_1 W_t + \beta_2 M_t + \beta_3 ST_t + \beta_4 SPD_t}}{1 + exp^{\beta_0 + \beta_1 W_t + \beta_2 M_t + \beta_3 ST_t + \beta_4 SPD_t}} \tag{6}$$

Actually, V_t serves as a hidden process setting $X_{t+1} = X_t$ or $X_{t+1} = Y_{t+1}$ at time $t + 1$. i.e.,

$$X_{t+1} = \begin{cases} X_t & \text{, with chance of } \theta_t \\ Y_{t+1} & \text{, with chance of } 1 - \theta_t. \end{cases}$$

The transition probability matrix is $\begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix}$ with

$$P_{ij} = \begin{cases} (1 - \pi) + \pi\theta_t & , \quad (i, j) = (0, 0) \\ \pi(1 - \theta_t) & , \quad (i, j) = (0, 1) \\ (1 - \pi)(1 - \theta_t) & , \quad (i, j) = (1, 0) \\ \pi(1 - \theta_t) + \theta_t & , \quad (i, j) = (1, 1). \end{cases}$$

This model specification guarantees that X_t has a stationary marginal distribution, i.e., $X_t \backsim Ber(\pi)$ for $\forall\ t$. Moreover, the one-step ahead prediction by this hidden Markov Model is:

$$X_{t+1}^t \triangleq \mathrm{E}(X_{t+1}|X_t) = \pi + \theta_t(X_t - \pi) \tag{7}$$

And the variance of the one-step ahead prediction error is:

$$\mathrm{Var}\,(X_{t+1} - X^{t}_{t+1}) = (1 - \theta_t^2)\pi(1 - \pi) \tag{8}$$

The joint probability of $\{X_s\}_{s=0}^{t}$ is as follow:

$$P(X_0, X_1, ..., X_t) = P(X_0)\prod_{s=1}^{t} P(X_s|X_{s-1}) \tag{9}$$

where $P(X_0) = \pi^{X_0}(1 - \pi)^{1-X_0}$ (stationary marginal of X_0).

Chang et al., (1984) proposed a similar hidden Markov model named the DAR(1) model in measuring the sequence of wet and dry days obtained from daily precipitation time series. In the DAR(1) model, θ_t is homogeneous i.e., $\theta_t = \theta$. However, in our proposed hidden Markov model, θ_t is heterogeneous depending on the financial indicators.

$$\log\left(\frac{\theta_t}{1 - \theta_t}\right) = \beta_0 + \beta_1 W_t + \beta_2 M_t + \beta_3 ST_t + \beta_4 SPD_t \tag{10}$$

Constrained Maximum Likelihood Estimation (MLE) method is used to estimate $(\underset{\sim}{\hat{\beta}}, \pi)$. i.e., $(\underset{\sim}{\hat{\beta}}, \hat{\pi}) = \underset{\underset{\sim}{\beta},\pi}{\arg\max} P(X_0, X_1, ..., X_t)$ subjected to $0 \leq \hat{\pi} \leq 1$, where $\underset{\sim}{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4)'$.

4. SIMULATION STUDY

A simulation study is designed to estimate and compare the power of the proposed hidden Markov model and the standard probit model constructed by Estrella and Mishkin (1996). In both the hidden Markov model and the standard probit model, we consider two explanatory variables in our numerical study, namely M_1 and M_2, which are independently simulated from $\mathcal{N}(0.5, 1)$.

In order to check the accuracy of our estimates, we consider five different sets of true values of parameters, $(\pi, \beta_0, \beta_1, \beta_2)'$. The true values of π are set close to 0.1095, the historical proportion of recession signals. And the true values of $(\beta_0, \beta_1, \beta_2)'$ are designed to encompass different combinations of positive and negative numbers. For each set of the values, we simulate 100 data point, $(M_{1,t}, M_{2,t}, Y_t)$, t=1, 2, ..., 100, following the procedures as below:

1. Draw $Y_t \stackrel{\text{iid}}{\sim} Ber(\pi)$, t=1, 2, ..., 100.
2. Compute θ_t using $\log\left(\frac{\theta_t}{1-\theta_t}\right) = \beta_0 + \beta_1 M_{1,t} + \beta_2 M_{2,t}$. And draw $V_t \stackrel{\text{iid}}{\sim} Ber(\theta_t)$, t=0, 1, ..., 99.
3. Compute X_{t+1} iteratively using $X_{t+1} = V_t X_t + (1 - V_t) Y_{t+1}$, t=0, 1, ..., 99.

The following is a list of combinations of true values of parameters:

1. $\pi = 0.10, \beta_0 = -2.00, \beta_1 = 1.00, \beta_2 = 2.50$;
2. $\pi = 0.11, \beta_0 = 3.00, \beta_1 = -0.50, \beta_2 = 1.00$;
3. $\pi = 0.12, \beta_0 = 3.50, \beta_1 = 5.00, \beta_2 = -2.00$;
4. $\pi = 0.13, \beta_0 = -0.30, \beta_1 = -2.80, \beta_2 = -1.40$;
5. $\pi = 0.14, \beta_0 = 2.60, \beta_1 = 1.20, \beta_2 = 0.50$.

The one-step ahead prediction based on the fitted models is $\hat{X}^t_{t+1} = \hat{\pi} + \hat{\theta}_t(X_t - \hat{\pi})$ with the variance $(1 - \hat{\theta}_t^2)\hat{\pi}(1 - \hat{\pi})$. All programs for simulation are coded using R 2.11.1.

Table 1 shows the mean and the standard error of $\hat{\pi}, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ of the hidden Markov model and the probit model. The true value of the parameters are set to: $\pi = 0.10, \beta_0 = -2.00$, $\beta_1 = 1.00, \beta_2 = 2.50$.

Table 1: Simulation 1 on Hidden Markov Model and Probit Model

Parameter	π	β_0	β_1	β_2
True Value	0.10	−2.00	1.00	2.50
Estimates:				
H.M. Model	0.0999	−2.2217	1.0967	2.7871
Std. Error	(0.0015)	(0.0640)	(0.0319)	(0.0634)
Probit Model	—	−1.1398	0.5687	1.4236
Std. Error	—	(0.0801)	(0.0565)	(0.0847)

The numerical results show that the estimates of the hidden Markov model are close to the true values of the parameters. While the estimates of the probit model have a large bias with a much larger standard error, meanwhile, their 95% confidence intervals fail to capture the true value of the parameters.

Table 2: Simulation 2 on Hidden Markov Model and Probit Model

Parameter	π	β_0	β_1	β_2
True Value	0.11	3.00	−0.50	1.00
Estimates:				
H.M. Model	0.1074	3.0781	−0.4914	1.2407
Std. Error	(0.0053)	(0.0735)	(0.0441)	(0.0584)
Probit Model	—	1.6426	−0.2418	0.4914
Std. Error	—	(0.0895)	(0.0710)	(0.0780)

Table 2 shows the mean and the standard error of $\hat{\pi}, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ of the hidden Markov model and the probit model. The true value of the parameters are set to: $\pi = 0.11, \beta_0 = 3.00$, $\beta_1 = -0.50, \beta_2 = 1.00$.

The numerical results show that the estimates of the hidden Markov model are close to the true values of the parameters. While the estimates of the probit model have a large bias with a much larger standard error, meanwhile, their 95% confidence intervals fail to capture the true value of the parameters.

Table 3 shows the mean and the standard error of $\hat{\pi}, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ of the hidden Markov model and the probit model. The true value of the parameters are set to: $\pi = 0.12, \beta_0 = 3.50$, $\beta_1 = 5.00, \beta_2 = -2.00$.

Table 3: Simulation 3 on Hidden Markov Model and Probit Model

Parameter	π	β_0	β_1	β_2
True Value	0.12	3.50	5.00	−2.00
Estimates:				
H.M. Model	0.1187	3.8342	5.5209	−2.3127
Std. Error	(0.0025)	(0.1217)	(0.2324)	(0.1114)
Probit Model	—	1.8965	2.7130	−1.0824
Std. Error	—	(0.1552)	(0.2100)	(0.1150)

The numerical results show that the estimates of the hidden Markov model are close to the true values of the parameters. While the estimates of the probit model have a large bias with a much larger standard error, meanwhile, their 95% confidence intervals fail to capture the true value of the parameters.

Table 4: Simulation 4 on Hidden Markov Model and Probit Model

Parameter	π	β_0	β_1	β_2
True Value	0.13	−0.30	−2.80	−1.40
Estimates:				
H.M. Model	0.1289	−0.3537	−3.2118	−1.5899
Std. Error	(0.0013)	(0.0353)	(0.13212)	(0.0533)
Probit Model	—	−0.1730	−1.5511	−0.7754
Std. Error	—	(0.0629)	(0.1002)	(0.0720)

Table 4 shows the mean and the standard error of $\hat{\pi}, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ of the hidden Markov model and the probit model. The true value of the parameters are set to: $\pi = 0.13, \beta_0 = -0.30$, $\beta_1 = -2.80, \beta_2 = -1.40$.

The numerical results show that the estimates of the hidden Markov model are close to the true values of the parameters. While the estimates of the probit model have a large bias with a much larger standard error, meanwhile, their 95% confidence intervals fail to capture the true value of the parameters.

Table 5 shows the mean and the standard error of $\hat{\pi}, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ of the hidden Markov model and the probit model. The true value of the parameters are set to: $\pi = 0.14, \beta_0 = 2.60$, $\beta_1 = 1.20, \beta_2 = 0.50$.

Table 5: Simulation 5 on Hidden Markov Model and Probit Model

Parameter	π	β_0	β_1	β_2
True Value	0.14	2.60	1.20	0.50
Estimates:				
H.M. Model	0.1487	2.6787	1.2927	0.4725
Std. Error	(0.0064)	(0.0721)	(0.0732)	(0.0675)
Probit Model	—	1.4393	0.5931	0.2444
Std. Error	—	(0.0743)	(0.0780)	(0.0711)

The numerical results show that the estimates of the hidden Markov model are close to the true values of the parameters. While the estimates of the probit model have a large bias with a much larger standard error, meanwhile, their 95% confidence intervals fail to capture the true value of the parameters.

Overall, the estimates and the standard error of the hidden Markov model and the probit model are provided in the simulation study. The results of the simulation study indicate that, compared to the probit model, the estimates of the hidden Markov model are closer to the true values of parameters. However, the estimates of the probit model have a larger standard error and the true value of the parameters are not captured by the 95% confidence interval.

5. DATA DESCRIPTION

The data examined in the standard probit model by Estrella and Mishkin (1996) are quarterly data for four explanatory variables: the percentage difference between the yield on a 10-year Treasury bond and a 3-month Treasury bill (Yield Curve Spread); the percentage change in the leading index of Conference Board (Lead); monetary growth (Money); and the percentage in the Standard and Poor's 500 index of stock price (Stock). As in Dueker (1997), we test the same set of explanatory variables. The difference is that we use monthly time series data ranging from January 1977 to July 2012 with 428 values total. All these values are generated by the records from the Federal Reserve of St. Louis. The binary values of 0 or 1 represent the signal of recessions occurring in certain periods are retrieved from the National Bureau of Economic Research (NBER). The composition of the variables are presented in the following subsections.

5.1. Explanatory Variables in the Hidden Markov Model

Time Periods: January 1977 to September 2012, 428 values in total.

Data Frequency: Monthly.

Yield Curve (W): The spread of yield between 10-year Treasury bond and 3-month Treasury bill.

Change in Yield Curve (ΔW): Change in the spread of yield between 10-year Treasury bond and 3-month Treasury bill.

Money (M): Monetary growth measured using M2 Money Stock.

Stock (ST): Change in the Standard and Poor's 500 index of stock prices.

Spread (SPD): The difference between the 6-month Certificated of Deposit and 6-month Treasury bill.

Change in Spread (ΔSPD): Change in the spread of yield between 6-month Certificated of Deposit and 6-month Treasury bill.

Recession Signals (X): Binary variable using 1 or 0 to indicate whether or not the economy is in recessions. The historical long-term proportion of recession signals is 0.1095.

5.2. Definition of Components

TB10: 10-Year Treasury Constant Maturity Rate

TB3: 3-Month Treasury Bill: Secondary Market Rate

M2: M2 Money Stock

CPI: Consumer Price Index for All Urban Consumers

S&P500: S&P 500 Stock Price Index

CP: 6-Month Certificate of Deposit : Secondary Market Rate

Bill: 6-Month Treasury Bill: Secondary Market Rate

5.3. Construction of Explanatory Variables

W_t	$\log(1 + (TB10_t/100)) - \log(1 + (TB3_t/100))$
ΔW_t	$W_t - W_{t-1}$
M_t	$\log(M2_t/CPI_t) - \log(M2_{t-1}/CPI_{t-1})$
ST_t	$\log(S\&P500_t/S\&P500_{t-1})$
SPD_t	$\log(1 + (CP_t/100)) - \log(1 + (Bill_t/100))$
ΔSPD_t	$SPD_t - SPD_{t-1}$

6. HISTORICAL DATA ANALYSIS AND RESULTS

6.1. Empirical Results

In this section, we apply our hidden Markov model and the probit model to the historical data of financial indicators and U.S. recession signals. To compare the predictive power of the two models, we partition the historical data into two data sets, namely, the training data set containing 348 observations taken from January 1977 to November 2005 and the testing data set containing 80 observations taken from November 2005 to July 2012. The purpose of the design is to let the training data set large enough to estimate the model coefficients and to let the testing data set big enough to contain the latest recession ranging from December 2007 to June 2009.

First, we estimate the model coefficients π and $\underset{\sim}{\beta}$ using the training data set. Then we apply the theoretical hidden Markov model and the standard probit model to forecast the probability of recessions during the period of November 2005 to July 2012 and compare our projection with the testing data.

Table 6: Estimates and Standard Error of Hidden Markov Model and Probit Model

Parameter	H.M. Model	Std. Error	Probit Model	Std. Error
π	0.1059	0.0041	—	—
β_0	1.5673	0.0823	−2.7673	0.3146
β_1	47.1223	2.5273	22.3333	9.1336
β_2	−119.8318	4.5229	52.7830	22.0639
β_3	−1.0241	0.5282	−0.3872	2.6761
β_4	4.7986	4.8512	145.1778	23.4663

Table 6 indicates $\hat{\pi} = 0.1059$ meaning the long-term proportion of recession signals estimated from the training data is 0.1059. Recoded by NBER, the observed proportion of recession signals is 0.1095. Therefore, our estimate of π is close to the observed proportion.

Table 6 also indicates that the probability of a change in economic status, either from non-recession to recession or vice versa is:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 1.57 + 47.12W_t - 119.83M_t - 1.02ST_t + 4.80SPD_t \tag{11}$$

The one-step ahead prediction based on our hidden Markov model is:

$$X_{t+1}^{t} = \pi + \theta_t(X_t - \pi) \tag{12}$$

X_{t+1}^{t} is the chance of occurring a recession signal in economy based on the information up to time t, where t=1, 2, ..., 80, t=1 represents November 2005 and t=80 represents July 2012.

Hense, given $\hat{\pi}$ and $\hat{\theta}_t$, our one-step ahead prediction is:

$$\hat{X}_{t+1}^{t} = \hat{\pi} + \hat{\theta}_t(X_t - \hat{\pi}) \tag{13}$$

The probit regression is generated from the following estimated coefficients:

$$P(R_t = 1) = \Phi(-2.77 + 22.33W_t + 52.78M_t - 0.39ST_t + 145.18SPD_t) \tag{14}$$

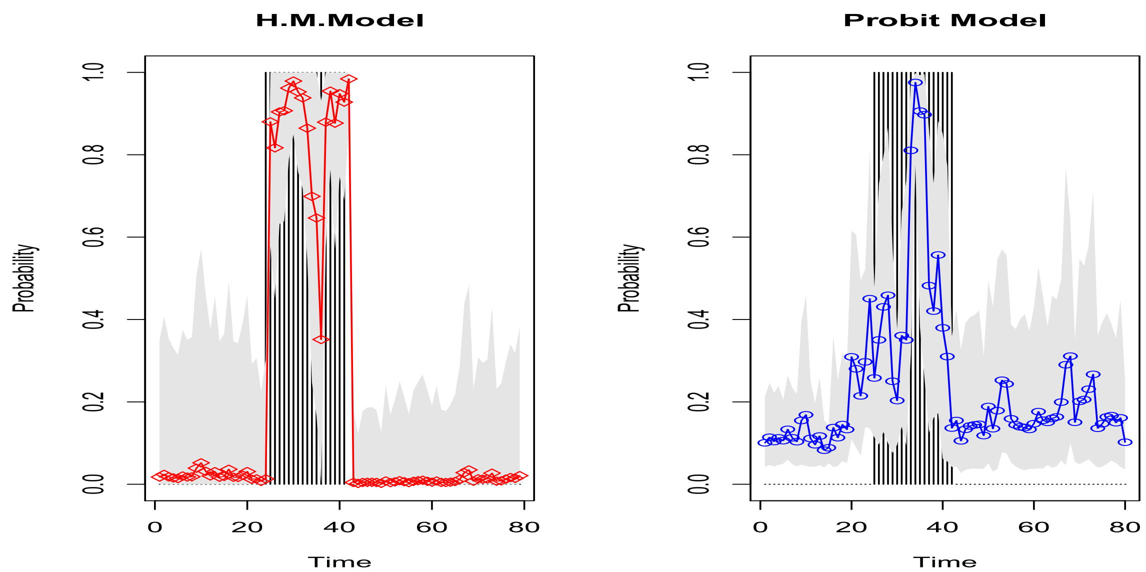

Figure 2: Recession Probabilities

According to the NBER, the latest recession started from December 2007, lasting eighteen months until May 2009. Figure 2 shows the projections of the theoretical hidden Markov model and the probit model. The shaded areas represent the 95% confidence interval of the projection.

The projection of the recession signals by the theoretical hidden Markov model in Figure 2 clearly indicates that the probability of recession increases significantly from nearly 0% to 88% in December 2007. And the probability of recessions decreases dramatically from 98% to nearly 0% in May 2009. Although the predicted recession probability falls down from May 2008 to November 2008 then rebounds sharply, the probability still over 25% threshold defined by Estrella and Mishkin (1996).

Meanwhile, the projection of the recession signals by the probit model in Figure 2 indicates that the probability of recession has slightly more waves prior to the 2007-09 recession from October 2007 to April 2008. Then the probability rises significantly, reaches the peak in August 2008, and drops down until it falls below the 25% threshold in December 2008. Namely, the probit model provides a general trend of recession but fails to capture all of the recession periods.

In short summary, Figure 2 indicates that both models with four explanatory variables have ability in predicting the recession with one month forecast horizon, while the hidden Markov model is more accurate compared to the probit model and roughly captures the signals of recession at the beginning and ending period.

Table 7: SS_E and MS_E of Hidden Markov Model and Probit Model

Criterion	H.M. Model	Probit Model
SS_E	2.7301	8.0487
MS_E	0.0350	0.1032

Table 7 provides the sum squared error (SS_E) and mean squared error (MS_E) for each model as criteria in measuring goodness of fit. The error is defined as the difference between the predicted value and the true value. In a general sense, the model with smaller SS_E and MS_E is better. The results indicate that the SS_E of the hidden Markov model is 2.7301, and it is relatively smaller than 8.0487 of the probit model. Meanwhile, the MS_E of the hidden Markov model is 0.0350, meaning there is only a 3.5% chance that the recession occurring in certain periods but the hidden Markov model fails to forecast. And this value is much smaller compared to 0.1032, which is the MS_E of the probit model.

In conclusion, the predicted recession probability figures and the SS_E and MS_E table indicate that both the hidden Markov model and the probit model have the ability in predicting the signals of recession in the U.S. However, the hidden Markov successfully captures all of the recession periods. Furthermore, the hidden Markov model makes an improvement by providing an estimate of long-term proportion of recession signals. Therefore, the hidden Markov model shows better predicting ability than the standard probit model in forecasting the recessions in the U.S.

6.2. Model Selection

The main result so far is that the hidden Markov model with four explanatory variables contains important information in forecasting U.S. recessions. In order to provide the evidence of potential usefulness of yield curve and select the best hidden Markov model, we perform the backward step-wise model selection process. The models with different combinations of explanatory variables we examined are:

1. $W + M + ST + SPD$;
2. $W + M + ST$;
3. $W + M$;
4. W;

We check the sum of squared error using the theoretical hidden Markov model with different combinations of financial indicators. Meanwhile, we analyze the predicted recession probability figure for each model.

Figure 3 illustrates the probability of recessions generated from the hidden Markov model with four explanatory variables: W, M, ST, and SPD. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 88% in December 2007. And the probability of recessions decreases dramatically from 98% to nearly 0% in May 2009. However, predicted recession probability drops from 98% in May 2008 to 35% in November 2008 and then rebounds to 98% sharply.

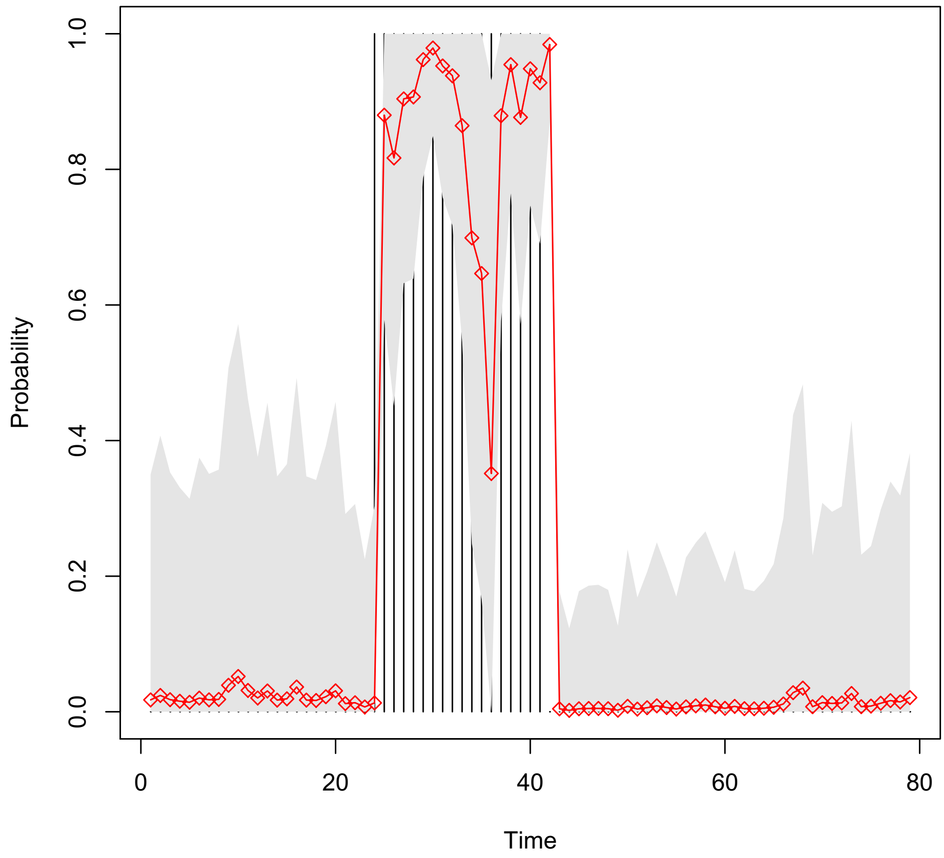

Figure 3: Model with Financial Indictors: W, M, ST, and SPD

Table 8: Estimates of Hidden Markov Model with W, M and ST

Parameter	H.M. Model	Std. Error
π	0.1079	0.0035
β_0	1.6356	0.0399
β_1	44.2374	2.1858
β_2	−113.2905	4.3627
β_3	−0.7954	0.5051

Table 8 indicates that the probability of Y_t equal to 1 is $\pi = 0.1079$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 1.64 + 44.24W_t - 113.29M_t - 0.80ST_t \tag{15}$$

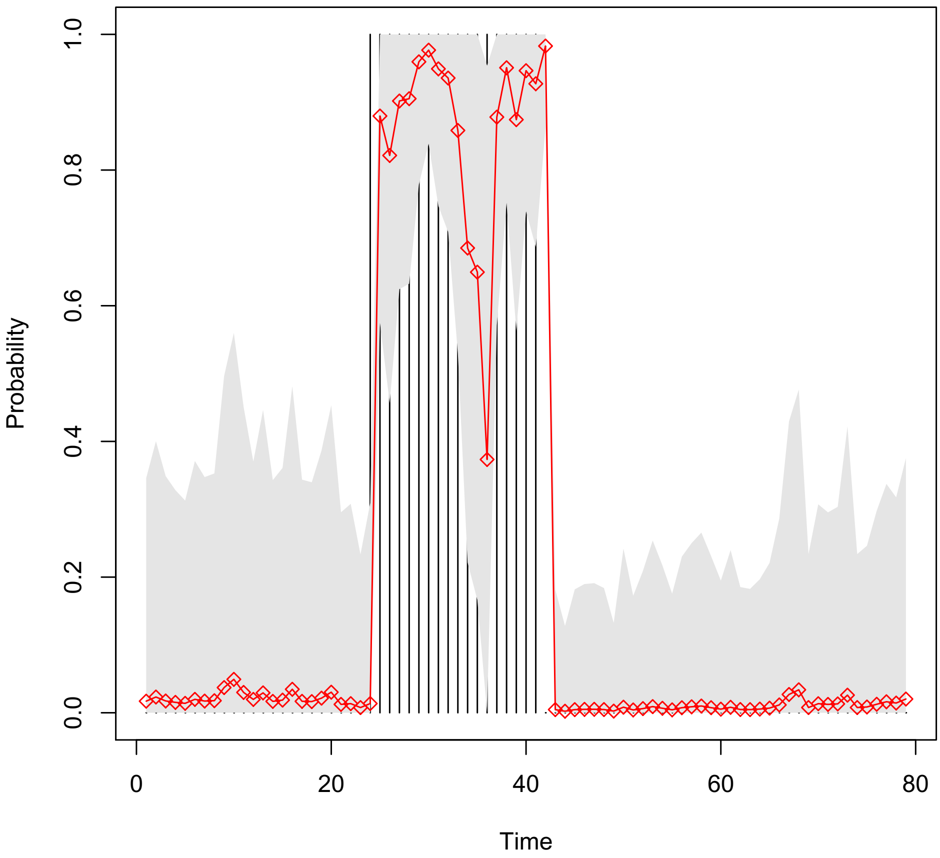

Figure 4: Model with Financial Indictors: W, M, and ST

Figure 4 illustrates the probability of recessions generated from the hidden Markov model with three explanatory variables: W, M, and ST. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 88% in December 2007. And the probability of recessions decreases dramatically from 98% to nearly 0% in May 2009. However, predicted recession probability drops from 98% in May 2008 to 38% in November 2008 and then rebounds to 95% sharply.

Table 9 indicates the probability of Y_t equals to 1 is $\pi = 0.1090$, and the probability of V_t equals to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 1.65 + 44.11W_t - 116.24M_t \quad (16)$$

Table 9: Estimates of Hidden Markov Model with W and M

Parameter	H.M. Model	Std. Error
π	0.1090	0.0035
β_0	1.6491	0.0393
β_1	44.1076	2.1141
β_2	-116.2361	4.3364

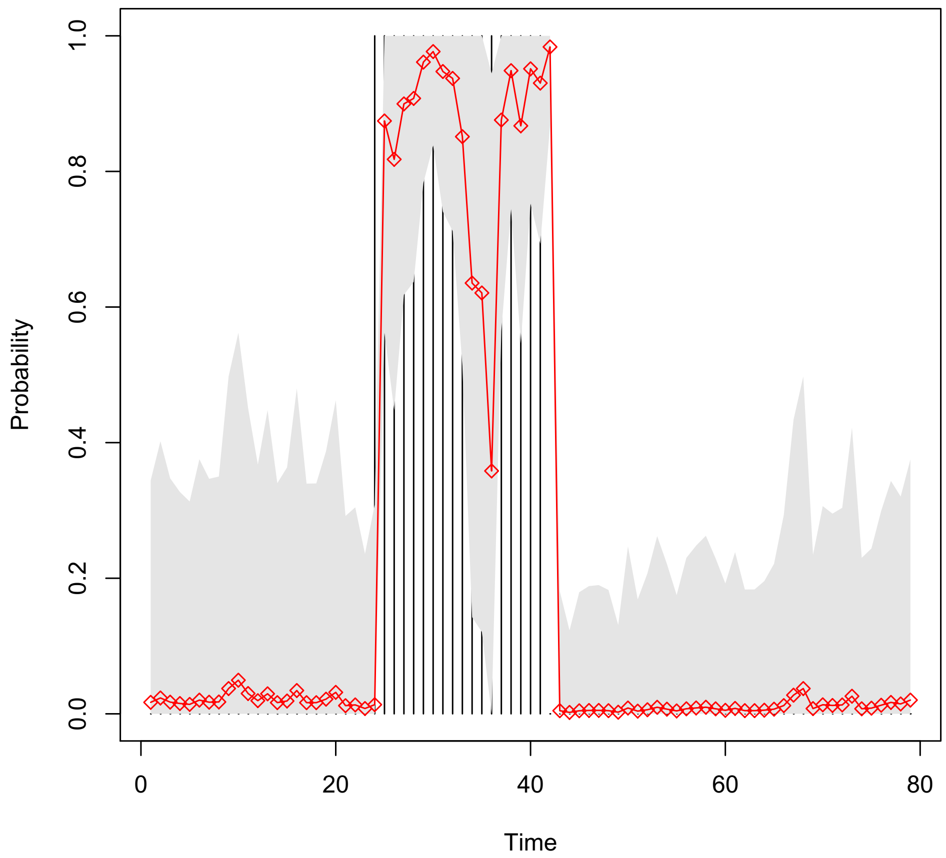

Figure 5: Model with Financial Indictors: W, and M

Figure 5 illustrates the probability of recessions generated from the hidden Markov model with two explanatory variables: W and M. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 82% in January 2008. And the probability of recessions decreases dramatically from 96% to nearly

0% in June 2009. However, predicted recession probability drops from 97% in May 2008 to 36% in November 2008 and then rebounds to 95% sharply.

Table 10: Estimates of Hidden Markov Model with W

Parameter	H.M. Model	Std. Error
π	0.1045	0.0006
β_0	1.7132	0.0066
β_1	17.7921	0.3117

Table 10 indicates that the probability of Y_t equal to 1 is $\pi = 0.1045$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 1.71 + 17.79W_t \tag{17}$$

Figure 6 illustrates the probability of recessions generated from the hidden Markov model with only one explanatory variable: W. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 87% in December 2007. And the probability of recessions decreases dramatically from 92% to nearly 0% in May 2009. During the recession period, the probability fluctuates slightly within a narrow range without significant shock.

Table 11: Statistics of Fitted Hidden Markov Model

Model	SS_E	MS_E	-2Log-Lik
$W + M + ST + SPD$	2.7372	0.0351	65.8994
$W + M + ST$	2.7015	0.0346	65.9050
$W + M$	2.7751	0.0356	65.9076
W	1.9876	0.0255	68.1917

Table 11 indicates the sum squared error, mean squared error and the -2Log-likelihood value for each model. The difference between the -2Log-Lik value of the full model with W, M, ST and SPD, and the model with W, M and ST is 0.0056, which is smaller than $\chi^2_{1,0.95} = 3.84$. Therefore, the full hidden Markov model can be reduced to the model with

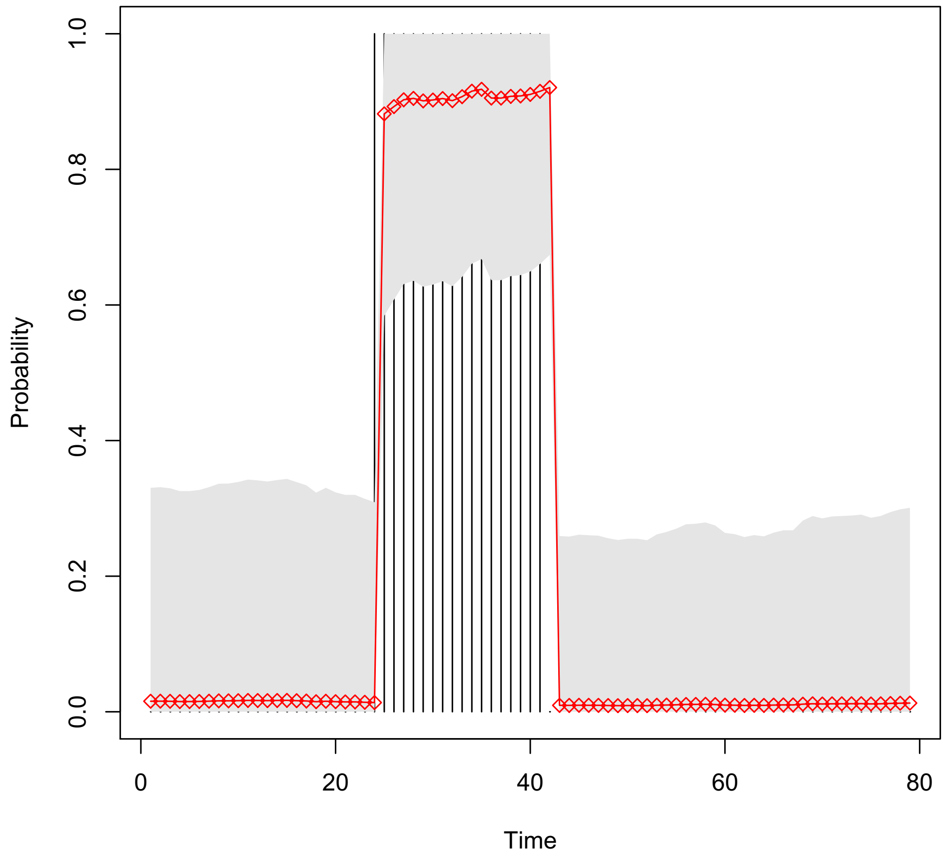

Figure 6: Model with Financial Indictor: W

three variables: W, M and ST. Moreover, the difference between the -2Log-Lik value of the model with W, M, and ST, and the model with W and M is 0.0026, which is also smaller than $\chi^2_{1,0.95} = 3.84$. Therefore, the full hidden Markov model can be further reduced to the model with two variables: W and M. Finally, the difference between the -2Log-Lik value of the model with W and M, and the model with the single variable W is 2.2841. Again, this value is smaller than $\chi^2_{1,0.95} = 3.84$. Therefore, the full hidden Markov model can be finally reduced to the model with the single explanatory variable, W, representing the yield curve spread.

In the hidden Markov model, θ_t, which is generated from log-link function of financial indicators, represents the probability of change in the recession signals. Since the definition of explanatory variables, M and ST, involves time change, we consider the change in W (ΔW) and the change in SPD (ΔSPD) in the further model selection process.

Table 12: Estimates of Hidden Markov Model with ΔW, M, ST, and ΔSPD

Parameter	H.M. Model	Std. Error
π	0.1484	0.0051
β_0	2.6030	0.0362
β_1	−120.4603	5.1189
β_2	−85.9299	3.9069
β_3	−0.2147	0.5023
β_4	−155.1789	10.6286

Table 12 indicates that the probability of Y_t equal to 1 is $\pi = 0.1484$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 2.60 - 120.46\Delta W_t - 85.93M_t - 0.21ST_t - 155.18\Delta SPD \tag{18}$$

Figure 7 illustrates the probability of recessions generated from the hidden Markov model with four explanatory variables: ΔW, M, ST, and ΔSPD. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 98% in December 2007. And the probability of recessions decreases dramatically from 97% to nearly 0% in May 2009. However, predicted recession probability drops from 98% in April 2008 to 31% in September 2008 and then rebounds to 96% sharply.

Table 13: Estimates of Hidden Markov Model with ΔW and M

Parameter	H.M. Model	Std. Error
π	0.1483	0.0052
β_0	2.5585	0.0352
β_1	−79.1756	3.7659
β_2	−86.1751	3.9959

Table 13 indicates that the probability of Y_t equal to 1 is $\pi = 0.1483$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 2.56 - 79.18\Delta W_t - 86.18M_t \tag{19}$$

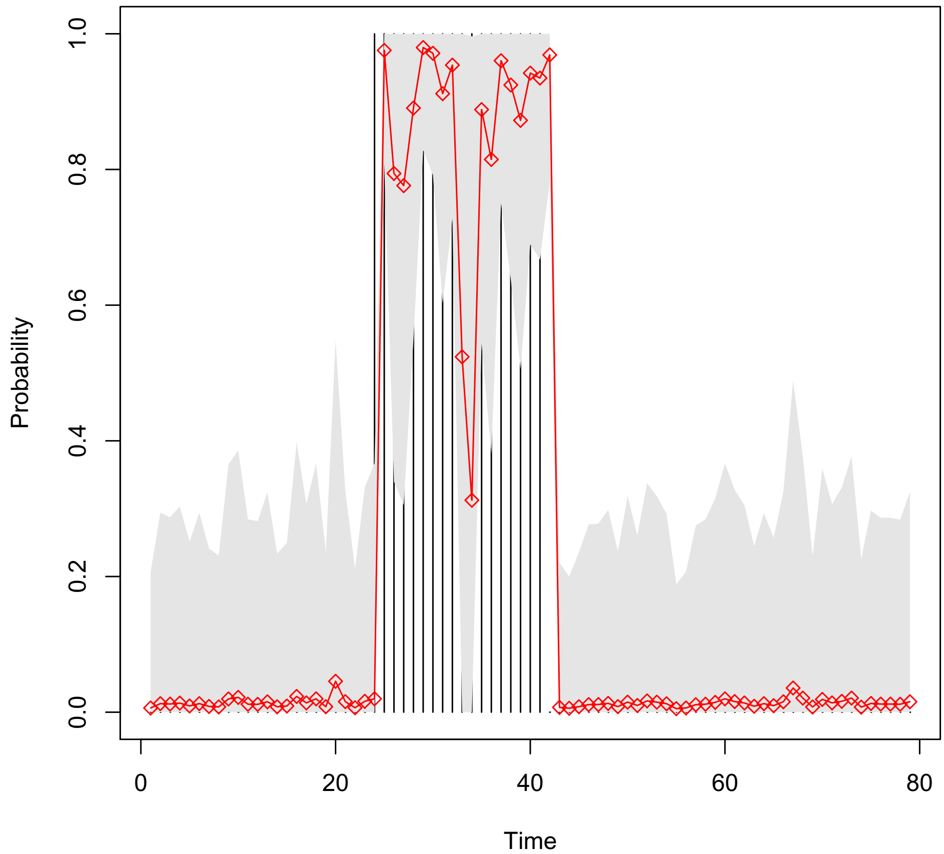

Figure 7: Model with Financial Indictors: ΔW, M, ST, and ΔSPD

Figure 8 illustrates the probability of recessions generated from the hidden Markov model with two explanatory variables: ΔW and M. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 93% in December 2007. And the probability of recessions decreases dramatically from 93% to nearly 0% in May 2009. However, predicted recession shakes down to 61% in September 2008 and smoothly rebounds to 93% in January 2009.

Table 14 indicates that the probability of Y_t equal to 1 is $\pi = 0.1284$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 2.20 - 63.05\Delta W_t \tag{20}$$

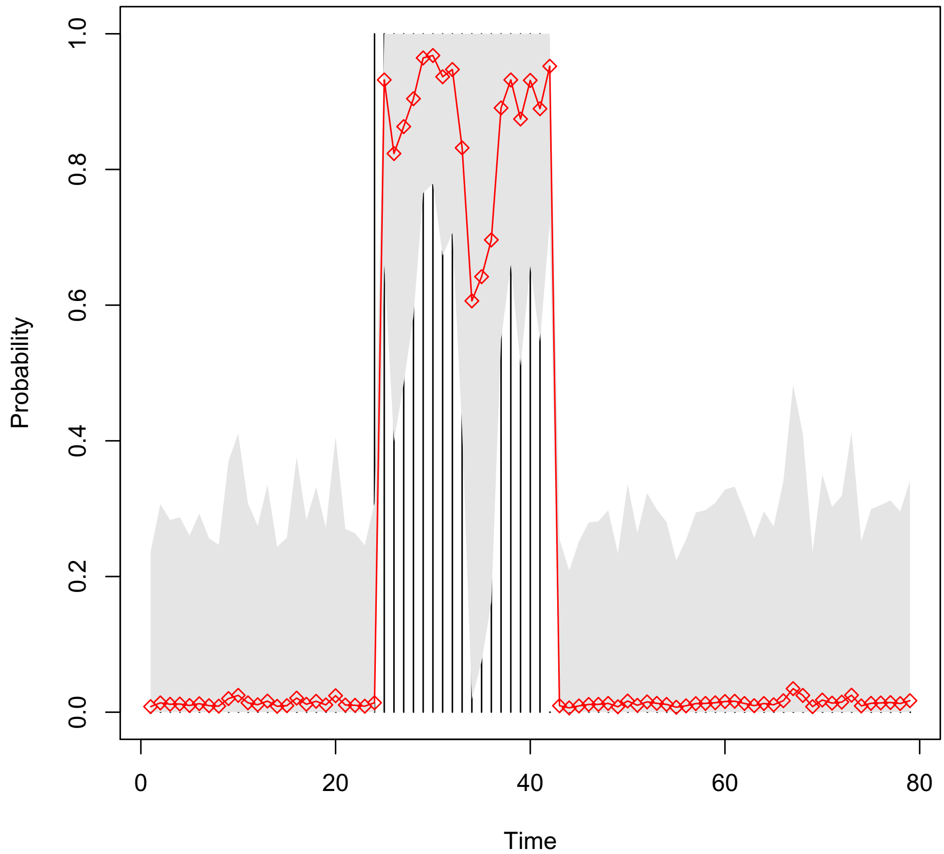

Figure 8: Model with Financial Indictors: ΔW and M

Table 14: Estimates of Hidden Markov Model with ΔW

Parameter	H.M. Model	Std. Error
π	0.1284	0.0034
β_0	2.2048	0.0209
β_1	−63.0468	3.0081

Figure 9 illustrates the probability of recessions generated from the hidden Markov model with one explanatory variable: ΔW. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 92% in December 2007. And the probability of recessions decreases dramatically from 89% to nearly 0% in May 2009. Different from the previous figures, the projection of the model with only ΔW does

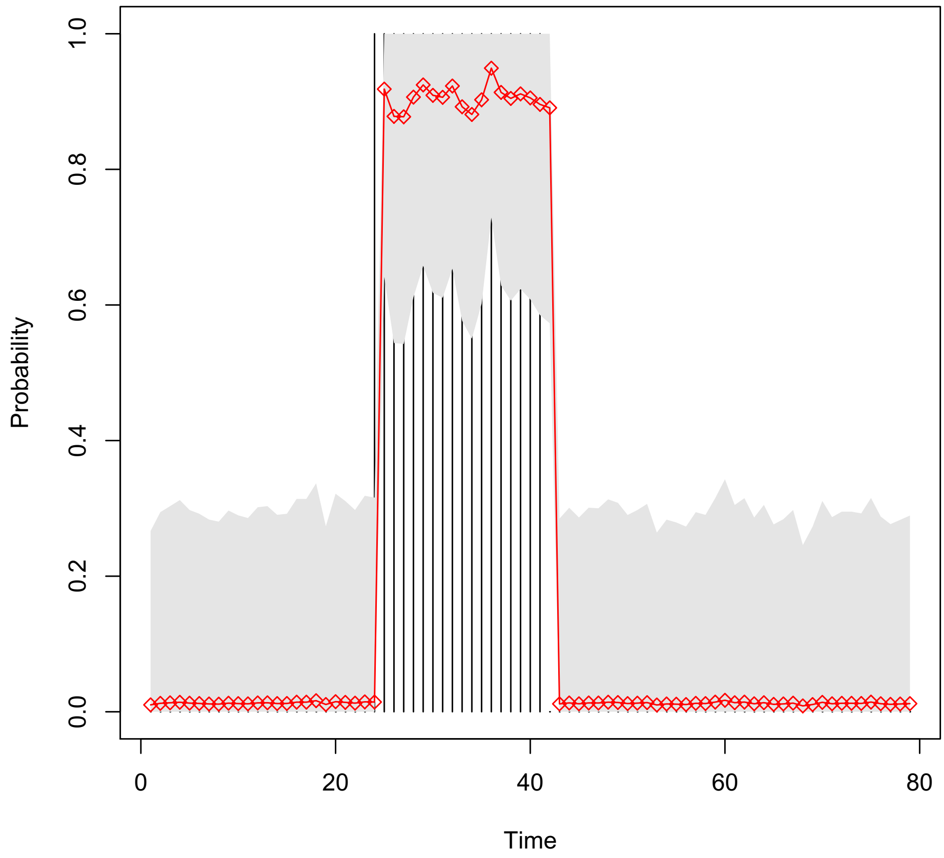

Figure 9: Model with Financial Indictor: ΔW

not have significant drops during the recession periods.

Table 15: Estimates of Hidden Markov Model with M

Parameter	H.M. Model	Std. Error
π	0.1155	0.0030
β_0	2.2564	0.0210
β_1	−78.8347	3.2630

Table 15 indicates that the probability of Y_t equal to 1 is $\pi = 0.1155$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 2.26 - 78.83M_t \tag{21}$$

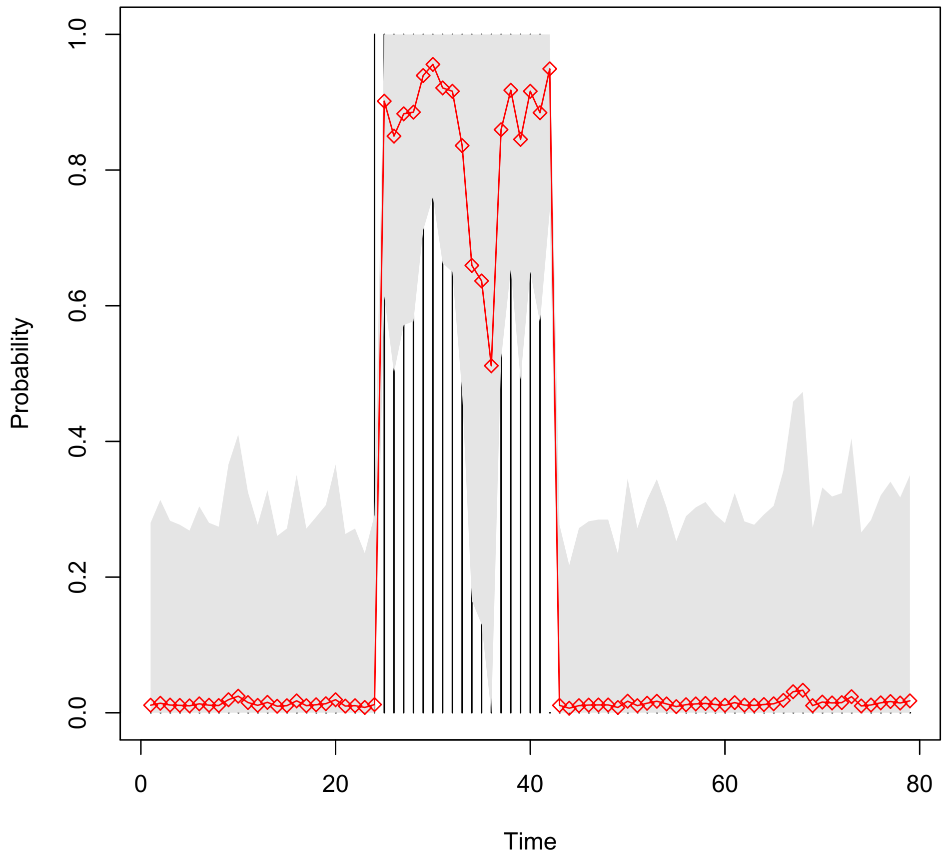

Figure 10: Model with Financial Indictor: M

Figure 10 illustrates the probability of recessions generated from the hidden Markov model with one explanatory variable: M. The projection of the recession signals indicates that the probability of recessions increases significantly from nearly 0% to 90% in December 2007. And the probability of recessions decreases dramatically from 95% to nearly 0% in May 2009. During the recession periods, the predicted recession probability drops from 96% in May 2008 to 51% in November 2008 and then rebounds to 92% sharply.

Table 16 indicates that the probability of Y_t equal to 1 is $\pi = 0.1048$, and the probability of V_t equal to 1 is transformed from:

$$\log\left(\frac{\hat{\theta}_t}{1-\hat{\theta}_t}\right) = 1.99 \tag{22}$$

Table 16: Estimates of Null Model

Parameter	Null Model	Std. Error
π	0.1048	0.0028
β_0	1.9893	0.0209

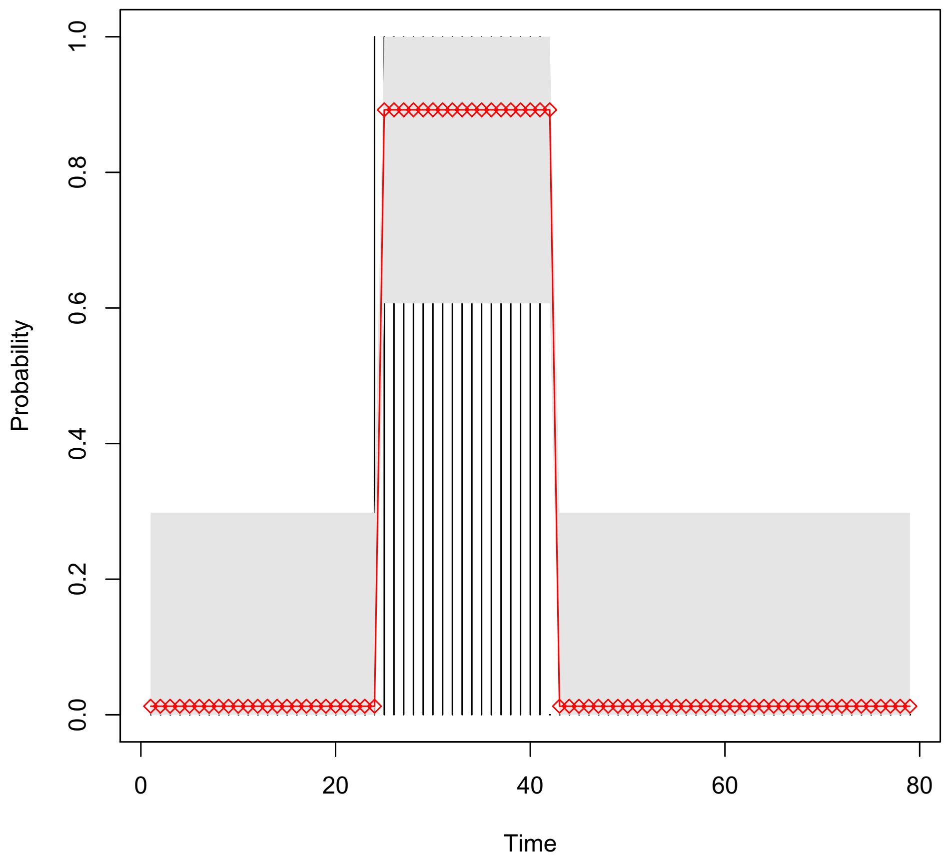

Figure 11: The Null Model

Figure 11 illustrates the probability of recessions generated from the null model. Since the null model does not involve financial indicators, θ_t in the null model is homogeneous. Therefore the projection of the recession probability of the null model increases from 0.01% to 89% in the December 2007 and remains constantly until May 2009.

Table 17: Statistics of Theoretical Hidden Markov Model

Model	BIC	AICc	SS_E	MS_E
$\Delta W + M + ST + \Delta SPD$	94.2476	75.1919	2.8069	0.0360
$\Delta W + M + ST$	89.3315	74.0631	2.5154	0.0322
$\Delta W + M + \Delta SPD$	88.3704	73.1019	2.9302	0.0376
$\Delta W + ST + \Delta SPD$	89.8446	74.5762	2.2550	0.0289
$M + ST + \Delta SPD$	90.5568	75.2883	2.6916	0.0345
$\Delta W + M$	83.5151	72.0460	2.4172	0.0310
$\Delta W + ST$	84.9903	73.5211	1.9535	0.0250
$\Delta W + \Delta SPD$	83.9865	72.5173	2.3168	0.0297
$M + ST$	84.8646	73.3955	2.4105	0.0309
$M + \Delta SPD$	84.6635	73.1944	2.6672	0.0342
$ST + \Delta SPD$	86.0492	74.5800	1.9950	0.0256
ΔW	79.1218	71.4639	1.9304	0.0247
M	78.9442	71.2863	2.5519	0.0327
ST	80.2034	72.5455	1.9966	0.0256
ΔSPD	80.2249	72.5670	1.9766	0.0253
$Null$	74.3711	70.5362	1.9779	0.0254

Table 17 indicates the BIC, AICc, SS_E, and MS_E for all possible combinations of models. In a general sense, the model for which AICc is smallest represents the best approximation to the true model. Therefore, statistically, the null model with AICc=70.5362 is selected to be the best model in predicting recession signals. The reason behind is that our hidden Markov model provides an accurate estimation on the long-term proportion of recession signals. However, taking the economic factors into consideration, the AICc of model with ΔW is not much different from the AICc of the null model. Meanwhile, the model with ΔW has the smallest MS_E. Therefore, the yield curve spread remains an important financial indicator in predicting U.S. recessions.

7. CONCLUSION

In this paper, we propose a hidden Markov model to forecast U.S. recession signals with the yield curve spread. As in Estrella and Mishkin (1996), we examine the yield curve spread between the 10-year Treasury bond and 3-month Treasury bill. Also, we take other supportive financial indictors affecting economy activities into account. The empirical results reveal that the hidden Markov model with yield curve spread as an explanatory variable provides quite satisfactory predictions for economy recession signals.

In this work, we compare the hidden Markov model to the standard probit model proposed by Estrella and Mishkin (1996) and also considered by Dueker (1997) and Chauvet et al. (2005). The reason for proposing the hidden Markov model is that the recession is a binary time series value with its own autocorrelation. However, the standard probit model does not consider the temporal dependence of recession signals. Although Dueker (1997) tries to incorporate the temporal dependence, the proposed modified probit model seems somewhat unnatural.

The numerical results indicate that our hidden Markov model outperforms the probit model in terms of the mean square prediction error. Moreover, the recession probability figures show that the hidden Markov model generally captures all of the latest recession periods starting from December 2007 and lasting until May 2009. Compared to the general trend provided by the standard probit model, the hidden Markov is highly improved. Another improvement of the hidden Markov model is that it provides an estimation of the long-term proportion of recession signals by using the training data.

In order to choose the best predictors from four financial indicators, we perform the backward step-wise model selection process. We choose the -2Log-Likelihood value and MS_E as criteria in measuring the fitness of the model. The results of model selection conclude that the hidden Markov model with the yield curve spread as the single explanatory variable is the best model.

In the hidden Markov model, θ_t, which is generated from the financial indicators, represents the probability of a change in economic status, either from non-recession to recession or vice versa. Since the explanatory variables M and ST consider the time change, we involve the change of W and the change of SPD in the full model selection process. In this case, we choose AICc, BIC, and MS_E as criteria in measuring the fitness of the model. Statistically, the null model is the best model. However, taking the economic factors into consideration, the model with yield curve spread remains useful in predicting U.S. recessions.

The results indicate that our proposed hidden Markov model is more accurate than the standard probit model in predicting U.S. recessions. One of the greatest improvements is that the hidden Markov model considers the dependency of binary time series value. Another improvement is that the model provides accurate estimation on the long-term proportion of recession signals. The model selection process reveals that the yield curve spread remains the single dominant predictor in forecasting U.S. recessions.

8. REFERENCES

Akaike, H. (1974). A new look at statistical model identification. *IEEE Transactions on Automatic Control, 19*, 713-723.

Bernard, H., & Gerlach, S. (1998). Does the term structure predict recessions? *The international evidence. International Journal of Finance and Economics, 3*, 195-215.

Boulier, B. L., & Steckler, H. O. (2000). The term spread as a monthly cyclical indicator. *Economics Letters, 66*, 79-83.

Chang, T. L., Kavvas M. L., & Delleur J. W. (1984). Modeling of sequences of wet and dry days by autoregressive moving average process. *Journal of Climate and Applied Meteorology. 23*, 1367-1378.

Chauvet, M., & Potter, S. (2005). Forecasting recession using the yield curve. *Journal of Forecasting, 4*(2), 77-103.

Davis, E. P., & Fagan, G. (1997). Are financial spreads useful indicators of future inflation and output growth in EU countries? *Journal of Applied Econometrics, 12*, 701-714.

Dueker, M. J. (1997). Strengthening the case for the yield curve as a predictor of U.S. recessions. [PDF document]. *Federal Reserve Bank of St. Louis Review, 79*(2), 41-51. Retrieved from http://research.stlouisfed.org/publications/review/97/03/9703md.pdf

Dueker, M. J. (2002). Regime-dependent recession forecasts and the 2001 recession. [PDF document]. *Federal Reserve Bank of St. Louis Review, 84*(6), 29-36. Retrieved from http://research.stlouisfed.org/publications/review/02/11/Dueker.pdf

Estrella, A. (1998). A new measure of fit for equations with dichotomous dependent variables. *Journal of Business and Economic Statistics, 16*, 198-205.

Estrella, A. (2005). Why does the yield curve predict output and inflation? *The Economic Journal, 115*, 722-744.

Estrella, A., & Hardouvelis, G. A. (1991). The term structure as a predictor of real economic activity. *Journal of Finance, 46*, 555-576.

Florio, A. (2004). The asymmetric effects of monetary policy. *Journal of Economic Surveys, 84*(2), 409-426.

Haubrich J. G. (2006). Does the yield curve signal recession? [PDF document]. *Federal*

Reserve Bank of Cleveland Review, 23-28. Retrieved from http://www.clevelandfed.org/Research/commentary/2006/0415.pdf

Marcellino, M., Stock J. H., & Watson, M. W. (2006): A comparison of direct and iterated AR methods for forecasting macroeconomic time series. *Journal of Econometrics, 135*, 499-526.

Morgan, D. P. (1993). Asymmetric effects of monetary policy. [PDF document]. *Federal Reserve Bank of Kansas City Economic Review*, 21-33. Retrieved from http://research.stlouisfed.org/publications/review/04/09/Ravn.pdf

APPENDIX A. R CODE FOR SIMULATION STUDY

```
n<- 100
beta0<-  -2
beta1<- 1
beta2<- 2.5
pi<- 0.1
x1<- rnorm(n, 0.5, 1)
x2<- rnorm(n, 0.5, 1)
theta0<-exp(beta0+beta1*x1+beta2*x2)/(1+exp(beta0+beta1*x1+beta2*x2))
A<- matrix(rep(NA,400), ncol = 4)
for(l in 1:100){
epsilon<- rbinom(n+1,1,pi)
v<- dim(n)
y<- dim(n+1)
t<- dim(n)
y[1]<- rbinom(1,1,pi)
for (i in 1:n) {
v[i]<- rbinom(1,1,theta0[i])
y[i+1]<- y[i]*v[i]+(1-v[i])*epsilon[i+1]
t[i]<- y[i+1]+2*y[i]  }
co<- dim(4)
g<- function(w1,w2,co,I) {
theta<- exp(co[1]+co[2]*w1+co[3]*w2)/(1+exp(co[1]+co[2]*w1+co[3]*w2))
```

```
(1-co[4]+co[4]*theta)*(I==0)+co[4]*(1-theta)*(I==1)+(1-theta)*(1-
co[4])*(I==2)+(theta+co[4]*(1-theta))*(I==3)}
L<- function(co) {
sum<- 0
for (k in 1:n) {
sum<- sum + log(g(x1[k],x2[k],co,t[k])) }
return (-sum - (y[1]*log(co[4])+(1-y[1])*log(1-co[4])))  }
tp<- c(beta0, beta1, beta2, pi)
ini<- c(2.5,1.5,0.5,0.15)
L(tp)
L(ini)
h<- function(w1,w2,co,I) {
theta<- exp(co[1]+co[2]*w1+co[3]*w2)/(1+exp(co[1]+co[2]*w1+co[3]*w2))
c((co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3))*theta*(1-theta),
  (co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3))*theta*(1-theta)*w1,
  (co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3))*theta*(1-theta)*w2,
   -(1-theta)*(I==0)+(1-theta)*(I==1)-(1-theta)*(I==2)+(1-theta)*(I==3) )}
gh<- function(co) {
sum1<- c(0,0,0,0)
for (k in 1:n) {
sum1<- sum1 + h(x1[k],x2[k],co,t[k])/g(x1[k],x2[k],co,t[k]) }
return (-sum1 - c(0,0,0,(y[1]-co[4])/(co[4]*(1-co[4])))) }
gh(tp)
```

```
gh(ini)
est<- constrOptim(theta=ini, f=L, grad=gh, ui=rbind(c(0,0,0,1),c(0,0,0,-1)), ci=c(0,-1),
method="BFGS")
A[l,] <- est$par
        l <- l+1}
beta0.e <- c(mean(A[,1]))
beta1.e <- c(mean(A[,2]))
beta2.e <- c(mean(A[,3]))
pi.e <- c(mean(A[,4]))
SE1 <- dim(100)
sum1 <- 0
for (i in 1:100){
SE1[i] <- (A[i,1]-beta0.e)^2
sum1 <- sum1+SE1[i]}
MSE1 <- sum1/99
ste1 <- sqrt(MSE1)
SE2 <- dim(100)
sum2 <- 0
for (i in 1:100){
SE2[i] <- (A[i,2]-beta1.e)^2
sum2 <- sum2+SE2[i]}
MSE2 <- sum2/99
ste2 <- sqrt(MSE2)
```

```
SE3 <- dim(100)

sum3 <- 0

for (i in 1:100){

SE3[i] <- (A[i,3]-beta2.e)^2

sum3 <- sum3+SE3[i]}

MSE3 <- sum3/99

ste3 <- sqrt(MSE3)

SE4 <- dim(100)

sum4 <- 0

for (i in 1:100){

SE4[i] <- (A[i,4]-pi.e)^2

sum4 <- sum4+SE4[i]}

MSE4 <- sum4/99

ste4 <- sqrt(MSE4)

mod <- glm(theta0 ~ x1 + x2, family = binomial(link = "probit"))

summary(mod)
```

APPENDIX B. R CODE FOR EMPIRICAL STUDY

```
data.wm <-  read.table(file.choose(),header=TRUE, sep="\t"
                       ,col.names=c("Year", "TB10", "TB3M", "Bill", "M2", "NCD", "RES"))
data.sp.1 <- read.table(file.choose(),header=TRUE, sep="\t"
                       ,col.names=c("Date","Value"))
data.sp <- na.omit(data.sp.1)
str(data.sp)
yr <- strftime(data.sp$Date, "%y")
mo <- strftime(data.sp$Date, "%m")
dy <- strftime(data.sp$Date, "%d")
amt <- data.sp$Value
dd <- data.frame(yr, mo, dy, amt)
dd.agg <- aggregate(amt~mo+yr, dd, mean)
dd.matrix <- format(rbind(dd.agg[153:428,],dd.agg[1:152,]),digits=4)
a <- as.matrix(dd.matrix[,1], ncol=1)
b <- as.matrix(dd.matrix[,2], ncol=1)
SP500 <- as.matrix(as.numeric(dd.matrix[,3], ncol=1))
databind <- cbind(a, b, data.wm$TB10, data.wm$TB3M, data.wm$Bill, data.wm$M2
                 ,data.wm$NCD, SP500, data.wm$RES)
n <- dim(databind)[1]
month <- as.matrix(databind[,1], ncol=1)
year <- as.matrix(databind[,2], ncol=1)
TB10 <- as.matrix(as.numeric(databind[,3]), ncol=1)
```

```
TB3M <- as.matrix(as.numeric(databind[,4]), ncol=1)

Bill <- as.matrix(as.numeric(databind[,5]), ncol=1)

M2 <- as.matrix(as.numeric(databind[,6]), ncol=1)

NCD <- as.matrix(as.numeric(databind[,7]), ncol=1)

X <- as.matrix(as.numeric(databind[,9]), ncol=1)

W <- (log((1+TB10/100)/(1+TB3M/100)))

M <- rep(NA, 427)

for (i in 1:427)

        {M[i+1] <- (log(M2[i+1]/M2[i]))}

ST <- rep (0, 427)

for (i in 1:427)

        {ST[i+1] <- (log(SP500[i+1]/SP500[i]))}

SPD <- (log((1+NCD/100)/(1+Bill/100)))

d.matrix <- cbind(rep(1, 347), W[2:348], M[2:348], ST[2:348], SPD[2:348])

n <- 347

int <- c(2.3,2.5,0.4,1.5,2.7, 0.13)

beta0 <- int[1]

beta1 <- int[2]

beta2 <- int[3]

beta3 <- int[4]

beta4 <- int[5]

pi<- int[6]

Xbeta0 <- beta0+beta1*W[2:348]+beta2*M[2:348]+beta3*ST[2:348]+beta4*SPD[2:348]
```

```
theta0<-exp(Xbeta0)/(1+exp(Xbeta0))

hist1 <- hist(theta0, col="lightblue", xlab="Prob", ylab="Count")

t<- dim(n)

for (i in 1:n) {

t[i]<- X[i+1]+2*X[i]}

co <- dim(6)

g<- function(w1,w2,w3,w4,co,I) {

Xbeta<- co[1]+co[2]*w1+co[3]*w2+co[4]*w3+co[5]*w4

theta<- exp(Xbeta)/(1+exp(Xbeta))

(1-co[6]+co[6]*theta)*(I==0)+co[6]*(1-theta)*(I==1)+(1-theta)*(1-

co[6])*(I==2)+(theta+co[6]*(1-theta))*(I==3)}

L<- function(co) {

sum<- 0

for (k in 1:n) {

sum<- sum + log(g(W[2:348][k],M[2:348][k],ST[2:348][k],SPD[k],co,t[k])) }

return (-sum - (X[1]*log(co[6])+(1-X[1])*log(1-co[6])))  }

ini<- c(2.3,2.5,0.4,1.5,2.7, 0.13)

L(ini)

h<- function(w1,w2,w3,w4,co,I) {

theta<-

exp(co[1]+co[2]*w1+co[3]*w2+co[4]*w3+co[5]*w4)/(1+exp(co[1]+co[2]*w1+co[3]*w2+co[4]

*w3+co[5]*w4))

        c((co[6]*(I==0)-co[6]*(I==1)-(1-co[6])*(I==2)+(1-co[6])*(I==3) )*theta*(1-theta),
```

```
        (co[6]*(I==0)-co[6]*(I==1)-(1-co[6])*(I==2)+(1-co[6])*(I==3))*theta*(1-theta)*w1,
        (co[6]*(I==0)-co[6]*(I==1)-(1-co[6])*(I==2)+(1-co[6])*(I==3))*theta*(1-theta)*w2,
        (co[6]*(I==0)-co[6]*(I==1)-(1-co[6])*(I==2)+(1-co[6])*(I==3))*theta*(1-theta)*w3,
        (co[6]*(I==0)-co[6]*(I==1)-(1-co[6])*(I==2)+(1-co[6])*(I==3))*theta*(1-theta)*w4,
        -(1-theta)*(I==0)+(1-theta)*(I==1)-(1-theta)*(I==2)+(1-theta)*(I==3) )}
gh<- function(co) {
sum1<- c(0,0,0,0,0,0)
for (k in 1:n) {
sum1<- sum1 +
h(W[2:348][k],M[2:348][k],ST[2:348][k],SPD[k],co,t[k])/g(W[2:348][k],M[2:348][k],ST[2:348]
[k],SPD[k],co,t[k]) }
return (-sum1 - c(0,0,0,0,0,(X[1]-co[6])/(co[6]*(1-co[6])))) }
est<- constrOptim(theta=ini, f=L, grad=gh, ui=rbind(c(0,0,0,0,0,1),c(0,0,0,0,0,-1)), ci=c(0,-1),
method="BFGS", hessian=T)
est$par
gh(est$par)
L(est$par)
hessian <- est$hessian
infor <- solve(hessian)
SE<- sqrt(diag(infor))*(1/sqrt(n))
est0 <- est$par[1]
est1 <- est$par[2]
est2 <- est$par[3]
```

```
est3 <- est$par[4]
est4 <- est$par[5]
pi.est <- est$par[6]
Xbeta2 <- est0+est1*W[349:428]+est2*M[349:428]+est3*ST[349:428]+est4*SPD[349:428]
theta2 <- exp(Xbeta2)/(1+exp(Xbeta2))
X.new <- X[349:428]
result <- rep(NA, 80)
for (i in 1:80)
        {result[i+1] <- pi.est + theta2[i]*(X.new[i]-pi.est)
        i <- i+1}
upper <- rep(NA, 80)
for (i in 1:80)
        {upper[i+1] <- result[i+1]+1.96*sqrt((1-theta2[i]^2)*pi.est*(1-pi.est))
        if(upper[i+1]>1)
        {upper[i+1]=1}
        else upper[i+1]=upper[i+1]}
lower <- rep(NA, 80)
for (i in 1:80)
        {lower[i+1] <- result[i+1]-1.96*sqrt((1-theta2[i]^2)*pi.est*(1-pi.est))
        if(lower[i+1]<0)
        {lower[i+1]=0}
        else lower[i+1]=lower[i+1]}
```

```
dif<- dim(79)

for (i in 1:79){

dif[i+1]<- (result[i+1]-X.new[i+1])^2

i<- i+1}

SSE<- sum(dif[2:80])

MSE<- SSE/78

n <- dim(d.matrix)[1]

mod1 <- glm(X[2:348] ~ W[2:348] + M[2:348] + ST[2:348] + SPD[2:348]

                    ,family = binomial(link = "probit"), data.frame(databind)[2:428,])

summary(mod1)

confint(mod1)

estt0 <- mod1$coef[1]

estt1 <- mod1$coef[2]

estt2 <- mod1$coef[3]

estt3 <- mod1$coef[4]

estt4 <- mod1$coef[5]

Xbeta3 <- estt0+estt1*W[349:428]+estt2*M[349:428]+estt3*ST[349:428]+estt4*SPD[349:428]

prob <- exp(Xbeta3)/(1+exp(Xbeta3))

uu <-  confint(mod1)[1,2]+confint(mod1)[2,2]*W[349:428]+confint(mod1)[3,2]*M[349:428]+

confint(mod1)[4,2]*ST[349:428]+confint(mod1)[5,2]*SPD[349:428]

up <- exp(uu)/(1+exp(uu))

ll <-  confint(mod1)[1,1]+confint(mod1)[2,1]*W[349:428]+confint(mod1)[3,1]*M[349:428]+

confint(mod1)[4,1]*ST[349:428]+confint(mod1)[5,1]*SPD[349:428]
```

```
lo <- exp(ll)/(1+exp(ll))

oldpar <- par(no.readonly = TRUE)
par(mfrow=c(1,2))
plot( X.new[2:80], xlab="Time", ylab="Probability", type="h", col="black")
polygon(c(1:79,79:1), c(upper[2:80],rev(lower[2:80])), ylim=c(0,1), col = "gray90", border =
NA)
lines(result[2:80], type="o", pch=23, lty=1:3, col="red")
title(main="H.M.Model")
plot( X.new,  xlab="Time", ylab="Probability",type="h", col="black")
polygon(c(1:80,80:1), c(up,rev(lo)), ylim=c(0,1), col = "gray90", border = NA)
lines(prob, type="o", pch=21, lty=1:3, col="blue")
title(main="Probit Model")
par(oldpar)
```

APPENDIX C. R CODE FOR MODEL SELECTION

```
data.wm <-  read.table(file.choose(),header=TRUE, sep="\t"
                              ,col.names=c("Year", "TB10", "TB3M", "Bill", "M2", "NCD", "RES"))
data.sp.1 <- read.table(file.choose(),header=TRUE, sep="\t"
                              ,col.names=c("Date","Value"))
data.sp <- na.omit(data.sp.1)
str(data.sp)
yr <- strftime(data.sp$Date, "%y")
mo <- strftime(data.sp$Date, "%m")
dy <- strftime(data.sp$Date, "%d")
amt <- data.sp$Value
dd <- data.frame(yr, mo, dy, amt)
dd.agg <- aggregate(amt~mo+yr, dd, mean)
dd.matrix <- format(rbind(dd.agg[153:428,],dd.agg[1:152,]),digits=4)
a <- as.matrix(dd.matrix[,1], ncol=1)
b <- as.matrix(dd.matrix[,2], ncol=1)
SP500 <- as.matrix(as.numeric(dd.matrix[,3], ncol=1))
databind <- cbind(a, b, data.wm$TB10, data.wm$TB3M, data.wm$Bill, data.wm$M2
                    ,data.wm$NCD, SP500, data.wm$RES)
n <- dim(databind)[1]
month <- as.matrix(databind[,1], ncol=1)
year <- as.matrix(databind[,2], ncol=1)
TB10 <- as.matrix(as.numeric(databind[,3]), ncol=1)
```

```
TB3M <- as.matrix(as.numeric(databind[,4]), ncol=1)

Bill <- as.matrix(as.numeric(databind[,5]), ncol=1)

M2 <- as.matrix(as.numeric(databind[,6]), ncol=1)

NCD <- as.matrix(as.numeric(databind[,7]), ncol=1)

X <- as.matrix(as.numeric(databind[,9]), ncol=1)

W <- (log((1+TB10/100)/(1+TB3M/100)))

M <- rep(NA, 427)

for (i in 1:427)

        {M[i+1] <- (log(M2[i+1]/M2[i]))}

ST <- rep (0, 427)

for (i in 1:427)

        {ST[i+1] <- (log(SP500[i+1]/SP500[i]))}

SPD <- (log((1+NCD/100)/(1+Bill/100)))

###########MODEL WITH W, M, ST #################

d.matrix <- cbind(rep(1, 347), W[2:348], M[2:348], ST[2:348])

n <- 347

int <- c(2.3,2.5,0.4,1.5, 0.13)

beta0 <- int[1]

beta1 <- int[2]

beta2 <- int[3]

beta3 <- int[4]

pi<- int[5]

Xbeta0 <- beta0+beta1*W[2:348]+beta2*M[2:348]+beta3*ST[2:348]
```

```
theta0<-exp(Xbeta0)/(1+exp(Xbeta0))

t<- dim(n)

for (i in 1:n) {

t[i]<- X[i+1]+2*X[i]}

co <- dim(6)

g<- function(w1,w2,w3,co,I) {

Xbeta<- co[1]+co[2]*w1+co[3]*w2+co[4]*w3

theta<- exp(Xbeta)/(1+exp(Xbeta))

(1-co[5]+co[5]*theta)*(I==0)+co[5]*(1-theta)*(I==1)+(1-theta)*(1-
co[5])*(I==2)+(theta+co[5]*(1-theta))*(I==3)}

L<- function(co) {

sum<- 0

for (k in 1:n) {

sum<- sum + log(g(W[2:348][k],M[2:348][k],ST[2:348][k],co,t[k])) }

return (-sum - (X[1]*log(co[5])+(1-X[1])*log(1-co[5])))  }

ini<- c(2.3,2.5,0.4,1.5, 0.13)

L(ini)

h<- function(w1,w2,w3,co,I) {

theta<-
exp(co[1]+co[2]*w1+co[3]*w2+co[4]*w3)/(1+exp(co[1]+co[2]*w1+co[3]*w2+co[4]*w3))

        c((co[5]*(I==0)-co[5]*(I==1)-(1-co[5])*(I==2)+(1-co[5])*(I==3) )*theta*(1-theta),

        (co[5]*(I==0)-co[5]*(I==1)-(1-co[5])*(I==2)+(1-co[5])*(I==3))*theta*(1-theta)*w1,

        (co[5]*(I==0)-co[5]*(I==1)-(1-co[5])*(I==2)+(1-co[5])*(I==3))*theta*(1-theta)*w2,
```

```
    (co[5]*(I==0)-co[5]*(I==1)-(1-co[5])*(I==2)+(1-co[5])*(I==3))*theta*(1-theta)*w3,
        -(1-theta)*(I==0)+(1-theta)*(I==1)-(1-theta)*(I==2)+(1-theta)*(I==3) )}
gh<- function(co) {
sum1<- c(0,0,0,0,0)
for (k in 1:n) {
sum1<- sum1 +
h(W[2:348][k],M[2:348][k],ST[2:348][k],co,t[k])/g(W[2:348][k],M[2:348][k],ST[2:348][k],co,t[
k]) }
return (-sum1 - c(0,0,0,0,(X[1]-co[5])/(co[5]*(1-co[5]))))   }
gh(ini)
L(ini)
est<- constrOptim(theta=ini, f=L, grad=gh, ui=rbind(c(0,0,0,0,1),c(0,0,0,0,-1)), ci=c(0,-1),
method="BFGS")
est$par
gh(est$par)
L(est$par)
est0 <- est$par[1]
est1 <- est$par[2]
est2 <- est$par[3]
est3 <- est$par[4]
pi.est <- est$par[5]
Xbeta2 <- est0+est1*W[349:428]+est2*M[349:428]+est3*ST[349:428]
theta2 <- exp(Xbeta2)/(1+exp(Xbeta2))
```

```
hist2 <- hist(theta2, col="lightblue", xlab="Prob", ylab="Count")
X.new <- X[349:428]
V <- rep(NA, 80)
Y <- rep(NA, 80)
a <- matrix(rep(NA, 800000), ncol=10000)
for (n in 1:10000)
{result <- rep(NA, 80)
for (i in 1:80)
        {Y[i+1] <- rbinom(1,1,pi.est)
        V[i] <- rbinom(1,1,theta2[i])
        result[i+1] <- V[i]*X.new[i]+(1-V[i])*Y[i+1]
        i <- i+1}
a[,n] <- result[2:81]
n <- n+1}
pred.X <- dim(80)
for (i in 1:80){
pred.X[i] <- mean(a[i,])}
dif<- dim(80)
for (i in 1:80){
dif[i]<- (pred.X[i]-X.new[i])^2
i<- i+1}
sum(dif)
sum(dif)/80
```

```
plot( X.new, xlab="Time", ylab="Probability", type="h", col="grey")
lines(pred.X, type="o", pch=23, lty=1:3, col="red")
###########MODEL WITH W, M, ST #####################
d.matrix <- cbind(rep(1, 347), W[2:348], M[2:348])
n <- 347
int <- c(2.3,2.5,0.4, 0.13)
beta0 <- int[1]
beta1 <- int[2]
beta2 <- int[3]
pi<- int[4]
Xbeta0 <- beta0+beta1*W[2:348]+beta2*M[2:348]
theta0<-exp(Xbeta0)/(1+exp(Xbeta0))
hist1 <- hist(theta0, col="lightblue", xlab="Prob", ylab="Count")
t<- dim(n)
for (i in 1:n) {
t[i]<- X[i+1]+2*X[i]}
co <- dim(6)
g<- function(w1,w2,co,I) {
Xbeta<- co[1]+co[2]*w1+co[3]*w2
theta<- exp(Xbeta)/(1+exp(Xbeta))
(1-co[4]+co[4]*theta)*(I==0)+co[4]*(1-theta)*(I==1)+(1-theta)*(1-
co[4])*(I==2)+(theta+co[4]*(1-theta))*(I==3)}
L<- function(co) {
```

```
sum<- 0
for (k in 1:n) {
sum<- sum + log(g(W[2:348][k],M[2:348][k],co,t[k])) }
return (-sum - (X[1]*log(co[4])+(1-X[1])*log(1-co[4])))   }
ini<- c(2.3,2.5,0.4, 0.13)     #initial value of parameters#
L(ini)
h<- function(w1,w2,co,I) {
theta<- exp(co[1]+co[2]*w1+co[3]*w2)/(1+exp(co[1]+co[2]*w1+co[3]*w2))
        c((co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3) )*theta*(1-theta),
        (co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3))*theta*(1-theta)*w1,
        (co[4]*(I==0)-co[4]*(I==1)-(1-co[4])*(I==2)+(1-co[4])*(I==3))*theta*(1-theta)*w2,
        -(1-theta)*(I==0)+(1-theta)*(I==1)-(1-theta)*(I==2)+(1-theta)*(I==3) )  }
gh<- function(co) {
sum1<- c(0,0,0,0)
for (k in 1:n) {
sum1<- sum1 + h(W[2:348][k],M[2:348][k],co,t[k])/g(W[2:348][k],M[2:348][k],co,t[k]) }
return (-sum1 - c(0,0,0,(X[1]-co[4])/(co[4]*(1-co[4])))) }
gh(ini)
est<- constrOptim(theta=ini, f=L, grad=gh, ui=rbind(c(0,0,0,1),c(0,0,0,-1)), ci=c(0,-1),
method="BFGS", hessian=T)
est$par
gh(est$par)
L(est$par)
```

```
hessian <- est$hessian

infor <- solve(hessian)

SE<- sqrt(diag(infor))*(1/sqrt(n))

SE

est0 <- est$par[1]

est1 <- est$par[2]

est2 <- est$par[3]

pi.est <- est$par[4]

Xbeta2 <- est0+est1*W[349:428]+est2*M[349:428]

theta2 <- exp(Xbeta2)/(1+exp(Xbeta2))

hist2 <- hist(theta2, col="lightblue", xlab="Prob", ylab="Count")

X.new <- X[349:428]

V <- rep(NA, 80)

Y <- rep(NA, 80)

a <- matrix(rep(NA, 800000), ncol=10000)

for (n in 1:10000)

{result <- rep(NA, 80)

for (i in 1:80)

        {Y[i+1] <- rbinom(1,1,pi.est)

        V[i] <- rbinom(1,1,theta2[i])

        result[i+1] <- V[i]*X.new[i]+(1-V[i])*Y[i+1]

        i <- i+1}

a[,n] <- result[2:81]
```

```
n <- n+1}
pred.X <- dim(80)
for (i in 1:80){
pred.X[i] <- mean(a[i,])}
dif<- dim(80)
for (i in 1:80){
dif[i]<- (pred.X[i]-X.new[i])^2
i<- i+1}
sum(dif)
sum(dif)/80
plot( X.new, xlab="Time", ylab="Probability" ,type="h", col="grey")
lines(pred.X, type="o", pch=23, lty=1:3, col="red")
###########MODEL WITH W ##########################
.matrix <- cbind(rep(1, 347), W[2:348])
n <- 347
int <- c(2.3,2.5, 0.13)
beta0 <- int[1]
beta1 <- int[2]
pi<- int[3]
Xbeta0 <- beta0+beta1*W[2:348]
theta0<-exp(Xbeta0)/(1+exp(Xbeta0))
hist1 <- hist(theta0, col="lightblue", xlab="Prob", ylab="Count")
t<- dim(n)
```

```
for (i in 1:n) {

t[i]<- X[i+1]+2*X[i]}

co <- dim(6)

g<- function(w1,co,I) {

Xbeta<- co[1]+co[2]*w1

theta<- exp(Xbeta)/(1+exp(Xbeta))

(1-co[3]+co[3]*theta)*(I==0)+co[3]*(1-theta)*(I==1)+(1-theta)*(1-

co[3])*(I==2)+(theta+co[3]*(1-theta))*(I==3)}

L<- function(co) {

sum<- 0

for (k in 1:n) {

sum<- sum + log(g(W[2:348][k],co,t[k])) }

return (-sum - (X[1]*log(co[3])+(1-X[1])*log(1-co[3])))  }

ini<- c(2.3,2.5, 0.13)     #initial value of parameters#

L(ini)

h<- function(w1,co,I) {

theta<- exp(co[1]+co[2]*w1)/(1+exp(co[1]+co[2]*w1))

        c((co[3]*(I==0)-co[3]*(I==1)-(1-co[3])*(I==2)+(1-co[3])*(I==3) )*theta*(1-theta),

        (co[3]*(I==0)-co[3]*(I==1)-(1-co[3])*(I==2)+(1-co[3])*(I==3))*theta*(1-theta)*w1,

        -(1-theta)*(I==0)+(1-theta)*(I==1)-(1-theta)*(I==2)+(1-theta)*(I==3) )}

gh<- function(co) {

sum1<- c(0,0,0)

for (k in 1:n) {
```

```
sum1<- sum1 + h(W[2:348][k],co,t[k])/g(W[2:348][k],co,t[k]) }

return (-sum1 - c(0,0,(X[1]-co[3])/(co[3]*(1-co[3]))))   }

gh(ini)

est<- constrOptim(theta=ini, f=L, grad=gh, ui=rbind(c(0,0,1),c(0,0,-1)), ci=c(0,-1),

method="BFGS", hessian=T)

est$par

gh(est$par)

L(est$par)

hessian <- est$hessian

infor <- hessian^(-1)

SE<- sqrt(diag(infor))*(1/sqrt(n))

SE

est0 <- est$par[1]

est1 <- est$par[2

pi.est <- est$par[3]

Xbeta2 <- est0+est1*W[349:428]

theta2 <- exp(Xbeta2)/(1+exp(Xbeta2))

hist2 <- hist(theta2, col="lightblue", xlab="Prob", ylab="Count")

X.new <- X[349:428]

V <- rep(NA, 80)

Y <- rep(NA, 80)

a <- matrix(rep(NA, 800000), ncol=10000)

for (n in 1:10000)
```

```
{result <- rep(NA, 80)
for (i in 1:80)
        {Y[i+1] <- rbinom(1,1,pi.est)
        V[i] <- rbinom(1,1,theta2[i])
        result[i+1] <- V[i]*X.new[i]+(1-V[i])*Y[i+1]
        i <- i+1}
a[,n] <- result[2:81]
n <- n+1}
pred.X <- dim(80)
for (i in 1:80){
pred.X[i] <- mean(a[i,])}
dif<- dim(80)
for (i in 1:80){
dif[i]<- (pred.X[i]-X.new[i])^2
i<- i+1}
sum(dif)
sum(dif)/80
plot( X.new, xlab)"Time", ylab="Probability", type="h", col="grey")
lines(pred.X, type="o", pch=23, lty=1:3, col="red")
```